

# Non-Relativistic Fluid Dual to Asymptotically AdS Gravity at Finite Cutoff Surface

Rong-Gen Cai\*, Li Li†, Yun-Long Zhang‡

Key Laboratory of Frontiers in Theoretical Physics,  
Institute of Theoretical Physics, Chinese Academy of Sciences,  
P.O. Box 2735, Beijing 100190, China

May 10, 2011

## Abstract

Using the non-relativistic hydrodynamic limit, we solve equations of motion for Einstein gravity and Gauss-Bonnet gravity with a negative cosmological constant within the region between a finite cutoff surface and a black brane horizon, up to second order of the non-relativistic hydrodynamic expansion parameter. Through the Brown-York tensor, we calculate the stress energy tensor of dual fluids living on the cutoff surface. With the black brane solutions, we show that for both Einstein gravity and Gauss-Bonnet gravity, the ratio of shear viscosity to entropy density of dual fluid does not run with the cutoff surface. The incompressible Navier-Stokes equations are also obtained in both cases.

## 1 Introduction

The AdS/CFT correspondence [1, 2, 3, 4] relates gravity in an anti-de Sitter (AdS) space-time to a strongly coupled conformal field theory (CFT) living on the boundary of the AdS space. Recently the AdS/CFT correspondence has been applied to various fields. By use of the AdS/CFT correspondence, one can calculate many quantities of strongly coupled CFTs through dual gravity theories. A remarkable example is the calculation of the ratio of shear viscosity to entropy density  $\eta/s$  of some field theories dual to the AdS Einstein gravity [5, 6, 7, 8]. In Einstein gravity, it was found that the ratio is a universal value

---

\*E-mail: cairg@itp.ac.cn

†E-mail: liliphy@itp.ac.cn

‡E-mail: zhangyl@itp.ac.cn

$1/4\pi$ , while in the case with  $R^2$  corrections [9, 10, 11, 12] there is a negative additional correction term. Furthermore, it was shown that  $\eta/s$  only depends on the value of effective coupling of transverse gravitons evaluated on the horizon [13, 14, 15, 16]. Under in the hydrodynamic limit, it was generally proven that some linear response coefficients are universal both in the AdS/CFT correspondence and membrane paradigm [14]. In particular, one can consider a fictitious membrane at a constant radial radius to express the AdS/CFT response in terms of the membrane paradigm language [14]. The dependence of the diffusion constant of dual fluid on the cutoff surface is interpreted as the Wilson renormalization group flow [17]. Some studies relating the radial radius of AdS space to energy scale of dual CFT appear in [18, 19, 20]. Some forms of holographic renormalization group flow equation independent of the cutoff were given recently in [21, 22, 23] and it was generally proven in [24] that they are actually equivalent to the radial evolution of the classical equation of motion [17].

Since the Wilson renormalization group flow theory does not require an ultraviolet completion of quantum field theory, the authors of [17] also do not insist on an asymptotically AdS region, instead they introduce a finite cutoff  $r_c$  outside the horizon in a general class of  $p + 2$ -dimensional black hole geometries. The dispersion relation of the gravitational fluctuations confined inside the cutoff is shown at long wavelengths to be that of a linearized  $p + 1$ -dimensional Navier-Stokes (NS) fluid living on the cutoff surface [17, 25, 26]. This remarkable relation was investigated in [25] that a given solution of the incompressible NS equations maps to a unique solution of the vacuum Einstein equations. An algorithm was presented in [26] for systematically reconstructing a solution for the  $p + 2$ -dimensional vacuum Einstein equations from a  $p + 1$ -dimensional fluid, to arbitrary order by extending the non-relativistic hydrodynamic expansion proposed in [25].

Clearly it is of great interest to develop a holography fluid description dual to an asymptotically flat gravitational configuration by introducing a finite cutoff. In this paper, however, we discuss this issue in asymptotically AdS gravity by introducing a finite radial cutoff, because when one takes the cutoff to be infinity, the dual field theory on the AdS boundary is well-defined and some results are comparable to those in the literatures. Clearly such a study could be a service to further discuss the case in asymptotically flat spacetimes.

Note that one can construct the stress energy tensor of the dual fluid order by order from the bulk gravity solution [27]. In this paper we follow the procedures in [14, 17, 25, 26], by introducing a finite cutoff surface  $\Sigma_c$  outside black hole horizon, generally discuss the effect of finite perturbations of the extrinsic curvature of  $\Sigma_c$  while keeping the intrinsic metric of the cutoff surface flat [25]. By applying two finite diffeomorphism transformations, in the non-relativistic hydrodynamic expansion limit we obtain black brane solutions, up to second order, of the non-relativistic hydrodynamic expansion parameter  $\epsilon$ , between the cutoff surface and the horizon in Einstein gravity with a negative cosmological constant. We calculate the stress energy tensor of the fluid on the cutoff surface. The results show that the ratio  $\eta/s$  is still  $1/4\pi$ , independent of the cutoff, which implies that it does not run along the radial coordinate. And it turns out that the stress energy tensor of the fluid obeys the incompressible Navier-Stokes equations. We also discuss the case of Gauss-

Bonnet gravity with a negative cosmological constant.

The paper is organized as follows. In Sec.(2) we introduce two finite diffeomorphism transformations to a general metric dual to fluid in flat spacetime, while keeping the induced metric of the cutoff surface invariant. We make the non-relativistic hydrodynamic expansion and solve gravitational equations to the second order of the expansion parameter. In Sec.(3) we apply this formulism to Einstein gravity with a negative cosmological constant. We find that the ratio of shear viscosity to entropy density of dual fluid on the finite cutoff surface  $\eta/s = 1/4\pi$ , independent of the cutoff and that the conservation equation of the stress energy tensor of the dual fluid gives an incompressible Navier-Stokes equation. In Sec.(4) we consider the case of Gauss-Bonnet gravity with a negative cosmological constant. The ratio of shear viscosity to entropy density is found to be  $\eta/s = (1 - 8\alpha)/4\pi$ , and corresponding incompressible Navier-Stokes equations are also obtained there. The conclusions and some discussions are included in Sec.(5).

## 2 Non-relativistic hydrodynamic expansion

To study the dynamics of fluid in  $p + 1$ -dimensional flat spacetime, we consider a generic  $p + 2$ -dimensional metric:

$$ds_{p+2}^2 = -h(r)d\tau^2 + 2d\tau dr + a(r)dx_idx^i, \quad (1)$$

where  $h(r)$  and  $a(r)$  are two functions of radial coordinate  $r$ . Introducing a finite cutoff surface  $\Sigma_c$  at  $r = r_c$  (outside black hole horizon if the horizon is present), the induced metric on the surface is flat

$$\gamma_{ab}dx^adx^b = -h(r_c)d\tau^2 + a(r_c)dx_idx^i, \quad (2)$$

where  $x^a \sim (\tau, x^i)$ . Introduce proper intrinsic coordinates  $\tilde{x}^a \sim (\tilde{\tau}, \tilde{x}^i)$  on  $\Sigma_c$  as

$$\tilde{x}^0 \equiv \tilde{\tau} = \sqrt{h(r_c)}\tau, \quad \tilde{x}^i = \sqrt{a(r_c)}x^i \quad (3)$$

the induced metric is simply given by

$$ds_{p+1}^2 = \eta_{ab}d\tilde{x}^ad\tilde{x}^b = -d\tilde{\tau}^2 + \delta_{ij}d\tilde{x}^id\tilde{x}^j. \quad (4)$$

In order to keep the intrinsic metric of  $\Sigma_c$  flat, following [26] we take two finite diffeomorphism transformations. The first one is a Lorentz boost with a constant boost parameter  $\beta_i$ . In the  $(\tilde{\tau}, \tilde{x}^i)$  coordinates, it is given by

$$\tilde{\tau} \rightarrow \gamma\tilde{\tau} - \gamma\beta_i\tilde{x}^i, \quad \tilde{x}^i \rightarrow \tilde{x}^i - \gamma\beta^i\tilde{\tau} + (\gamma - 1)\frac{\beta^i\beta_j}{\beta^2}\tilde{x}^j, \quad (5)$$

where

$$\tilde{u}^a = \gamma(1, \beta^i), \quad \gamma = \frac{1}{\sqrt{1 - \beta^2}}, \quad \beta_i = \delta_{ij}\beta^j \quad (6)$$

In the  $(\tau, x^i)$  coordinates, we have

$$u^a = \frac{(1, v^i)}{\sqrt{h(r_c) - v^2}}, \quad v^2 = v_i v^i = a(r_c) \delta_{ij} v^i v^j, \quad v^i \equiv \beta^i \sqrt{\frac{h(r_c)}{a(r_c)}}. \quad (7)$$

Thus we can get the boosted metric

$$\begin{aligned} ds_{p+2}^2 &= \frac{d\tau^2}{1 - v^2/h(r_c)} \left( -h(r) + \frac{a(r)}{a(r_c)} v^2 \right) + 2\gamma d\tau dr - \frac{2\gamma v_i}{h(r_c)} dx^i dr \\ &\quad + \frac{2v_i}{1 - v^2/h(r_c)} \left( \frac{h(r)}{h(r_c)} - \frac{a(r)}{a(r_c)} \right) dx^i d\tau \\ &\quad + \left[ a(r) \delta_{ij} - \frac{v_i v_j}{h(r_c)(1 - v^2/h(r_c))} \left( \frac{h(r)}{h(r_c)} - \frac{a(r)}{a(r_c)} \right) \right] dx^i dx^j. \end{aligned} \quad (8)$$

The second is a transformation of  $r$  and associated re-scalings of  $\tau$  and  $x^i$

$$r \rightarrow k(r), \quad \tau \rightarrow \tau \sqrt{\frac{h(r_c)}{h[k(r_c)]}}, \quad x^i \rightarrow x^i \sqrt{\frac{a(r_c)}{a[k(r_c)]}}, \quad (9)$$

where we consider the case with  $k(r)$  being a linear function of  $r$  as  $k(r) = br + c$ , with  $b$  and  $c$  two constants. In this case the general metric (1) becomes

$$ds_{p+2}^2 = -h[k(r)] \frac{h(r_c)}{h[k(r_c)]} d\tau^2 + 2b \sqrt{\frac{h(r_c)}{h[k(r_c)]}} d\tau dr + a[k(r)] \frac{a(r_c)}{a[k(r_c)]} dx_i dx^i. \quad (10)$$

When the solution describes a black brane, the cutoff surface  $r_c$  is required to be outside the horizon  $r_c > r_h$ . When  $h(r) = r$ ,  $a(r) = 1$ , the metric (1) just describes a flat space-time written in the ingoing Rindler coordinates, and the transformations (5) and (9) agree with those in [26] if we take  $k(r) = r - r_h$ .

After taking the two coordinate transformations one after another, the resulted metric still solves the corresponding gravitational field equations. But if we further promote  $v_i$  and  $\delta k(r) \equiv k(r) - r = (b - 1)r + c$  to be dependent on the coordinates  $x^a$ , (that is  $v_i, b, c$  are no longer constants), the transformed metric is no longer an exact solution of gravitational field equations. In order to solve the gravity equations of motion, we take the so-called hydrodynamics expansion and non-relativistic limit. Namely we will take the scaling

$$\partial_r \sim \epsilon^0, \quad \partial_i \sim \epsilon^1, \quad \partial_\tau \sim \epsilon^2 \quad i, j = 1, .. p \quad (11)$$

together with

$$v_i \sim \epsilon, \quad \delta k(r) \sim \epsilon^2, \quad (12)$$

where  $\epsilon$  will be viewed as an expansion parameter.

As  $r$  is arbitrary between  $r_h$  and  $r_c$ , we demand both  $(b - 1)$  and  $c$  scale as  $\epsilon^2$ . Then up to order  $\epsilon^2$ , one has

$$h[k(r)] = h(r) + h'(r)\delta k(r), \quad a[k(r)] = a(r) + a'(r)\delta k(r), \quad (13)$$

and the transformed metric changes to

$$\begin{aligned}
ds_{p+2}^2 = & -h(r)d\tau^2 + 2d\tau dr + a(r)dx_i dx^i \\
& - 2 \left( \frac{a(r)}{a(r_c)} - \frac{h(r)}{h(r_c)} \right) v_i dx^i d\tau - 2 \frac{v_i}{h(r_c)} dx^i dr \\
& + \left( \frac{a(r)}{a(r_c)} - \frac{h(r)}{h(r_c)} \right) \left[ v^2 d\tau^2 + \frac{v_i v_j}{h(r_c)} dx^i dx^j \right] + \frac{v^2}{h(r_c)} d\tau dr \\
& - h(r) \left( \frac{h'(r)\delta k(r)}{h(r)} - \frac{h'(r_c)\delta k(r_c)}{h(r_c)} \right) d\tau^2 + 2 \left( (b-1) - b \frac{h'(r_c)\delta k(r_c)}{2h(r_c)} \right) d\tau dr \\
& + a(r) \left( \frac{a'(r)\delta k(r)}{a(r)} - \frac{a'(r_c)\delta k(r_c)}{a(r_c)} \right) dx_i dx^i + O(\epsilon^3).
\end{aligned} \tag{14}$$

The first and second lines of metric (14) are of order  $\epsilon^0$  and  $\epsilon^1$  respectively, the other lines are all of order  $\epsilon^2$ . Note that if one takes

$$h(r) = r, \quad a(r) = 1 \quad k(r) = r - 2P, \tag{15}$$

and the corresponding non-relativistic scaling

$$v_i = v_i^\epsilon(\tau, x^i) = \epsilon v_i(\epsilon^2 \tau, \epsilon x^i), \quad P = P^\epsilon(\tau, x^i) = \epsilon^2 P(\epsilon^2 \tau, \epsilon x^i), \tag{16}$$

it is easy to see that the metric (14) is the same as the one in [25], up to order  $\epsilon^2$ .

Next we consider the general asymptotically AdS black brane solutions. Following [27], we take the AdS radius to be unit so that

$$h(r) = r^2 f(r), \quad a(r) = r^2, \tag{17}$$

where  $f(r)$  is an arbitrary function of  $r$ , but it will be given in (22) and (49) for Einstein gravity and Gauss-Bonnet gravity respectively. Consider  $k(r)$  as a following transformation

$$k(r) = r(1 - P) \Rightarrow \delta k(r) = -rP, \tag{18}$$

where  $P$  is a small parameter. As will be shown shortly, in fact, the parameter  $P$  multiplied by a factor  $\frac{r_c h'(r_c)}{2h(r_c)}$  is the pressure density of the dual fluid.

Substituting (17) and (18) in (14), the first line of the metric with different  $f(r)$  solves the equations of motion for the corresponding gravity with a negative cosmological constant exactly. The remainder terms in (14) could be treated as the perturbations of the metric, as  $(v_i, P)$  will turn out to be the small parameters of the dual non-relativistic fluid. To be more specific, we consider  $v_i = v_i(x^i, \tau)$  and  $P = P(x^i, \tau)$  depending on the coordinates  $x^a$ , but independent of  $r$ , and take the non-relativistic hydrodynamic limit in (16), as well as (11). Then the metric (14) only solves corresponding gravity equations with the cosmological constant  $\Lambda = -\frac{p(p+1)}{2}$  at order  $\epsilon^1$ . In order to solve the equations to the next order, we need to add correction terms to the metric [26, 27]. Let's consider the constraint equation first [27]. At order  $\epsilon^2$ , we find that there is only one nontrivial constraint condition:  $\partial_i v^i = 0$ .

This will turn out to be the incompressibility of the dual fluid. With this constraint condition, it turns out that at order  $\epsilon^2$ , the source terms only have tensor modes and only the following tensor correction terms need to be added to the metric:

$$\frac{r^2}{r_c^2} F(r) (\partial_i v_j + \partial_j v_i) dx^i dx^j \quad (19)$$

where  $F(r)$  is chosen to cancel the source terms at order  $\epsilon^2$  and to keep regular at the horizon. In order to keep the induced metric  $\gamma_{ab}$  invariant, we also need to choose the gauge such that  $F(r_c) = 0$  [26]. Then our final metric up to  $\epsilon^2$  is:

$$\begin{aligned} ds_{p+2}^2 = & -r^2 f(r) d\tau^2 + 2d\tau dr + r^2 dx_i dx^i \\ & - 2\frac{r^2}{r_c^2} \left(1 - \frac{f(r)}{f(r_c)}\right) v_i dx^i d\tau - 2\frac{v_i}{r_c^2 f(r_c)} dx^i dr \\ & + \frac{r^2}{r_c^2} \left(1 - \frac{f(r)}{f(r_c)}\right) \left[v^2 d\tau^2 + \frac{v_i v_j}{r_c^2 f(r_c)} dx^i dx^j\right] + \frac{v^2}{r_c^2 f(r_c)} d\tau dr \\ & + r^2 f(r) \left(\frac{rf'(r)}{f(r)} - \frac{r_c f'(r_c)}{f(r_c)}\right) P d\tau^2 + \frac{r_c f'(r_c)}{f(r_c)} P d\tau dr \\ & + \frac{r^2}{r_c^2} F(r) (\partial_i v_j + \partial_j v_i) dx^i dx^j + O(\epsilon^3), \end{aligned} \quad (20)$$

where the terms in last three lines are all of order  $\epsilon^2$ .

### 3 Fluid dual to Einstein Gravity

Consider Einstein gravity with a cosmological constant  $\Lambda = -\frac{p(p+1)}{2}$  in  $p+2$  dimensions, the equations of motion are given by <sup>1</sup>

$$E_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \frac{p(p+1)}{2} g_{\mu\nu} = 0. \quad (21)$$

The Einstein's field equations admit the asymptotically  $AdS_{p+2}$  black brane solution as

$$ds_{p+2}^2 = -r^2 f(r) d\tau^2 + 2d\tau dr + r^2 dx_i dx^i, \quad f(r) = 1 - \frac{r_h^{p+1}}{r^{p+1}}. \quad (22)$$

Another class of dynamical solutions we are interested here can also be found in the region between the cutoff surface  $\Sigma_c$  and the black brane horizon up to order  $\epsilon^2$ . The approach to find the solution is described in the previous section, it turns out that the corresponding  $F(r)$  term in (20) is <sup>2</sup>

$$F(r) = \int_r^{r_c} dx \left[ \left(1 - \frac{r_h^p}{x^p}\right) \frac{1}{x^2 f(x)} \right]. \quad (23)$$

<sup>1</sup>Here we use  $\{\mu, \nu, \dots\}$  to stand for the bulk spacetime indices.

<sup>2</sup>We have checked the form  $F(r)$  for the case of  $0 \leq p \leq 8$  by mathematic calculation, and we expect that this form is also valid for higher  $p$  [29].

Here the boundary condition that  $F'(r)$  is regular at  $r = r_h$  has been imposed. Additionally, the integral upper bound has been chosen to keep  $\gamma_{ab}$  invariant and this matches the result in [28] when we take the cutoff surface to infinity. With the constraint condition  $\partial_i v^i = 0$ , the metric (20) solves the Einstein's field equations (21) at order  $\epsilon^2$ , i.e.,  $E_{\mu\nu} = O(\epsilon^3)$ . In what follows, we will consider the  $p = 3$  case as a calculation example. The fixed boundary condition is just the invariant induced metric  $\gamma_{ab}$  on  $\Sigma_c$

$$\gamma_{ab} dx^a dx^b = -r_c^2 f(r_c) d\tau^2 + r_c^2 (dx_1^2 + dx_2^2 + dx_3^2). \quad (24)$$

With the gravity solution (20), one can calculate the stress energy tensor of dual fluid. The Brown-York stress energy tensor  $T_{ab}$  evaluated at the cutoff hyper-surface  $\Sigma_c$  is [17]

$$T_{ab} = \frac{1}{8\pi G} (\gamma_{ab} K - K_{ab} + C\gamma_{ab}), \quad (25)$$

where  $K_{ab}$  is the extrinsic curvature tensor of  $\Sigma_c$ ,  $K$  is its trace, and  $C$  is an ambiguous constant. It is obvious that  $T_{ab}^C = \frac{C}{8\pi G} \gamma_{ab}$  only have order  $\epsilon^0$  terms. After some straightforward calculations, we obtain the stress energy tensor of the dual fluid as <sup>3</sup>

$$T_{ab} = T_{ab}^{(0)} + T_{ab}^{(1)} + T_{ab}^{(2)} + O(\epsilon^3) \quad (26)$$

where

$$\left\{ \begin{array}{l} T_{ab}^{(0)} dx^a dx^b = -r_c^2 f_c \left( 6\sqrt{f_c} + 2C \right) d\tau^2 + r_c^2 \left( 6\sqrt{f_c} + 2C + \frac{r_c f'_c}{\sqrt{f_c}} \right) dx_i dx^i \\ T_{ab}^{(1)} dx^a dx^b = -2 \frac{r_c f'_c}{\sqrt{f_c}} v_i dx^i d\tau \\ T_{ab}^{(2)} dx^a dx^b = \frac{r_c f'_c}{\sqrt{f_c}} (3r_c^2 f_c P + v^2) d\tau^2 + \frac{r_c f'_c}{\sqrt{f_c}} \left[ \frac{v_i v_j}{r_c^2 f_c} + \left( 1 + \frac{r_c f'_c}{2f_c} \right) r_c^2 P \delta_{ij} \right] dx^i dx^j \\ \quad - \frac{1 + r_c^2 f_c F'_c}{r_c \sqrt{f_c}} (\partial_i v_j + \partial_j v_i) dx^i dx^j. \end{array} \right. \quad (27)$$

Since the conservation equations of the Brown-York stress energy tensor are just the Gauss-Codazzi formulas of Einstein's field equations [33, 17]. This means that the conservation equations of the dual fluid stress energy tensor are the corresponding constraint equations of gravity equations. With the conservation equations of the stress energy tensor, we have the first nontrivial equation at order  $\epsilon^2$ ,

$$\partial^a T_{a\tau} = \frac{r_c f'_c}{\sqrt{f_c}} \partial^i v_i = \frac{4r_h^4}{r_c^2 \sqrt{r_c^4 - r_h^4}} \partial_i v^i = 0 \Rightarrow \partial_i v^i = 0. \quad (28)$$

This agrees with the constraint equation in the gravity side and it turns out that the dual fluid is incompressible. Taking this to be the case, to the next order, we have

$$\partial^a T_{ai} = \frac{f'_c}{r_c f_c \sqrt{f_c}} \left[ \partial_\tau v_i - \frac{f_c (1 + r_c^2 f_c F'_c)}{f'_c} \partial^2 v_i + r_c^2 f_c \left( 1 + \frac{r_c f'_c}{2f_c} \right) \partial_i P + v^j \partial_j v_i \right] = 0. \quad (29)$$

---

<sup>3</sup>Here we use the units  $16\pi G = 1$  and  $f_c = f(r_c)$ ,  $f'_c = f'(r_c)$ ,  $F'_c = F'(r_c)$ ,  $\dots$  for short.

This would be the constraint equation of the gravity metric of order  $\epsilon^3$  and more correction terms need to be added to solve the equations of motion of gravity at this order.

In the infinity boundary limit  $r_c \rightarrow \infty$ , as the cutoff surface  $\Sigma_c$  is intrinsic flat, from the surface counter-term approach [30], we have to take  $C = -3$  for our  $AdS_5$  case in order to remove the divergence in the stress energy tensor. In AdS/CFT, the background metric  $h_{ab}$  for the fluid stress energy tensor  $\langle T_{ab} \rangle$  is redefined by stripping off the divergent conformal factor from the boundary metric [31]

$$h_{ab} = \lim_{r_c \rightarrow \infty} \frac{\gamma_{ab}}{r_c^2}, \quad \sqrt{-h} h^{ab} \langle T_{bc} \rangle = \lim_{r_c \rightarrow \infty} \sqrt{-\gamma} \gamma^{ab} T_{bc}. \quad (30)$$

Our result can also recover the result for dual fluid on the boundary in the non-relativistic limit [27, 28], we will see this later. To see clearly the properties of the dual fluid, it is more convenient to rewrite (27) in the  $(\tilde{\tau}, \tilde{x}^i)$  coordinates:

$$\left\{ \begin{array}{l} \tilde{T}_{ab}^{(0)} d\tilde{x}^a d\tilde{x}^b = - \left( 6\sqrt{f_c} + 2C \right) d\tilde{\tau}^2 + \left( 6\sqrt{f_c} + 2C + \frac{r_c f'_c}{\sqrt{f_c}} \right) d\tilde{x}_i d\tilde{x}^i \\ \tilde{T}_{ab}^{(1)} d\tilde{x}^a d\tilde{x}^b = -2 \frac{r_c f'_c}{\sqrt{f_c}} \beta_i d\tilde{x}^i d\tilde{\tau} \\ \tilde{T}_{ab}^{(2)} d\tilde{x}^a d\tilde{x}^b = \frac{r_c f'_c}{\sqrt{f_c}} (3P + \beta^2) d\tilde{\tau}^2 + \frac{r_c f'_c}{\sqrt{f_c}} \left[ \beta_i \beta_j + \left( 1 + \frac{r_c f'_c}{2f_c} \right) P \delta_{ij} \right] d\tilde{x}^i d\tilde{x}^j \\ \quad - (1 + r_c^2 f_c F'_c) (\tilde{\partial}_i \beta_j + \tilde{\partial}_j \beta_i) d\tilde{x}^i d\tilde{x}^j. \end{array} \right. \quad (31)$$

In general, the stress energy tensor of relativistic fluid in 4-dimensional Minkowski background  $\tilde{\gamma}_{ab} = \eta_{ab}$  can be written as [28, 34]

$$\tilde{T}_{ab} = \tilde{\rho} \tilde{u}_a \tilde{u}_b + \tilde{p} \tilde{h}_{ab} - 2\eta \tilde{\sigma}_{ab} - \zeta \tilde{\theta} \tilde{h}_{ab}, \quad (32)$$

where

$$\tilde{h}_{ab} = \tilde{u}_a \tilde{u}_b + \tilde{\gamma}_{ab}, \quad \tilde{\sigma}_{ab} = \frac{1}{2} \tilde{h}_{ad} \tilde{h}_{be} (\tilde{\partial}^d \tilde{u}^e + \tilde{\partial}^e \tilde{u}^d) - \frac{1}{3} \tilde{\theta} \tilde{h}_{ab}, \quad \tilde{\theta} = \tilde{\partial}_a \tilde{u}^a. \quad (33)$$

In the non-relativistic limit and with the incompressible condition, up to order  $\epsilon^2$ , we have

$$\tilde{T}_{\tau\tau} = \tilde{\rho} + (\tilde{\rho} + \tilde{p}) \beta^2, \quad \tilde{T}_{\tau i} = -(\tilde{\rho} + \tilde{p}) \beta_i, \quad \tilde{T}_{ij} = (\tilde{\rho} + \tilde{p}) \beta_i \beta_j + \tilde{p} \delta_{ij} - \eta (\tilde{\partial}_i \beta_j + \tilde{\partial}_j \beta_i). \quad (34)$$

Comparing our dual fluid stress energy tensor (31) with this form, we can get from  $\tilde{T}_{ab}^{(0)}$  in (31) the energy density and pressure of dual fluid at order  $\epsilon^0$  as

$$\rho_0 = -(6\sqrt{f_c} + 2C), \quad p_0 = 6\sqrt{f_c} + 2C + \frac{r_c f'_c}{\sqrt{f_c}}, \quad \omega_0 \equiv \rho_0 + p_0 = \frac{r_c f'_c}{\sqrt{f_c}}. \quad (35)$$

Further we can obtain from  $\tilde{T}_{ab}^{(2)}$  in (31) the  $r_c$  dependent dynamic viscosity

$$\eta_c \equiv \eta(r_c) = (1 + r_c^2 f_c F'_c) = \frac{r_h^3}{r_c^3} = \frac{1}{16\pi G} \frac{r_h^3}{r_c^3}. \quad (36)$$

As  $\partial_i \beta_j \sim \epsilon^2$ , we see the viscosity  $\eta_c$  is of order  $\epsilon^0$ , it is better to compare this with the background of the fluid entropy density, which is just the background black brane horizon entropy density. The entropy density  $s_c$  associated with the cutoff surface is [17]

$$s_c \equiv s(r_c) = \frac{1}{4G} \frac{r_h^3}{r_c^3} \implies \frac{\eta_c}{s_c} = \frac{1}{4\pi}. \quad (37)$$

We see that the ratio of shear viscosity to entropy density is independent of the cutoff  $r_c$ , which means that the ratio does not run with the cutoff. Note that for the static background solution (22), the Hawking temperature  $T_H$  of the horizon and the local temperature  $T_c$  on the cutoff surface respectively are

$$T_H = \frac{[r^2 f(r)]'}{4\pi} \Big|_{r=r_h} = \frac{r_h}{\pi}, \quad T_c = \frac{T_H}{\sqrt{r_c^2 f_c}} = \frac{1}{\sqrt{r_c^2 f_c}} \frac{r_h}{\pi}. \quad (38)$$

We see that the following thermodynamic relation still holds

$$\omega_0 = T_c s_c = \frac{r_c f'_c}{\sqrt{f_c}} = \frac{4r_h^4}{r_c^2 \sqrt{r_c^4 - r_h^4}}, \quad (39)$$

for the dual fluid on the cutoff surface. In addition, the dimensionless coordinate invariant diffusivity  $\bar{D}_c$  defined in [17] is found to be

$$\bar{D}_c \equiv T_c \frac{\eta_c}{\omega_0} = \frac{\eta_c}{s_c} = \frac{1}{4\pi}, \quad (40)$$

a universal constant.

Next let's read off in the  $(\tilde{\tau}, \tilde{x}^i)$  coordinates the energy density and pressure of the dual fluid, to order  $\epsilon^2$ ,

$$\rho_c = \rho_0 + 3\omega_0 P, \quad p_c = p_0 + \omega_0 \left( 1 + \frac{r_c f'_c}{2f_c} \right) P. \quad (41)$$

The Hamiltonian constraint on  $\Sigma_c$  would play a role analogous to that of the equation of state for a conventional fluid [26]. For the fluid dual to Einstein gravity with a negative cosmology constant  $\Lambda$ , if we define  $\hat{T}_{ab} = \tilde{T}_{ab} - 2C\gamma_{ab} = 2(\gamma_{ab}K - K_{ab})$ , the Hamiltonian constraint would turn out to be  $\hat{T}^2 - p\hat{T}_{ab}\hat{T}^{ab} + 8p\Lambda = 0$ . This constraint provides the relation between the energy density and pressure of the dual fluid. If we evaluate the Hamiltonian constraint using the Brown-York stress tensor (31), the first nontrivial equation is encountered at order  $\epsilon^2$ , which is just the incompressible condition  $\tilde{\partial}_i \beta^i = 0$ . Taking this to be the case and substituting (34) into the constraint, we can get  $(\tilde{\rho} + 2C)(\tilde{\rho} + 3\tilde{p} - 4C) + 72 = 0$  up to order  $\epsilon^2$ . It can be checked that the energy density and pressure read off from the Brown-York tensor of the dual fluid satisfy this equation of state, by use of  $(\rho_c, p_c)$  in (41).

We can also calculate the trace of the stress energy tensor (31), to order  $\epsilon^2$ ,

$$\tilde{T}_c \equiv \tilde{T}_{ab} \tilde{\gamma}^{ab} = 4 \left[ 3(f_c^{1/2} + f_c^{-1/2}) + 2C \right] + \frac{3\omega_0^2}{2\sqrt{f_c}} P, \quad (42)$$

which in general does not vanish. However, if take the cutoff surface  $r_c \rightarrow \infty$ , and consider the conformal invariance of dual field theory on the AdS boundary, namely,  $\tilde{T}_c \rightarrow 0$ , we can also recover the counterterm factor  $C = -3$ . In addition, we find that

$$\lim_{r_c \rightarrow \infty} \rho_c r_c^4 = 3r_h^4 (1 + 4P), \quad \lim_{r_c \rightarrow \infty} p_c r_c^4 = r_h^4 (1 + 4P). \quad (43)$$

Note that  $r_c^4$  in (43) should be absorbed by the dual fluid stress energy tensor  $\langle T_{ab} \rangle$  defined in the flat spacetime  $h_{ab}$  in (30), the results reproduce the holographic values.

Following [28], we define the pressure density as

$$P_c \equiv \frac{p_c - p_0}{\rho_0 + p_0} = \left(1 + \frac{r_c f'_c}{2f_c}\right) P \quad (44)$$

and the kinematic viscosity as

$$\nu_c \equiv \frac{\eta_c}{\omega_0} = \left(\frac{r_h}{r_c}\right)^3 \frac{\sqrt{f_c}}{4} \left(\frac{r_c}{r_h}\right)^4 = \frac{1}{4\pi T_c}. \quad (45)$$

Then we can rewrite the conservation equation (29) in the  $(\tilde{\tau}, \tilde{x}^i)$  coordinates with Minkowski metric  $\eta_{ab}$  and 3-velocity  $\beta_i$  as

$$\frac{1}{\omega_0} \tilde{\partial}^a \tilde{T}_{ai} = \tilde{\partial}_\tau \beta_i - \nu_c \tilde{\partial}^2 \beta_i + \tilde{\partial}_i P_c + \beta^j \tilde{\partial}_j \beta_i = 0. \quad (46)$$

This is nothing, but precisely the incompressible Navier-Stokes equation up to order  $\epsilon^3$  of the dual fluid in flat spacetime.

## 4 Fluid dual to Gauss-Bonnet Gravity

The Einstein-Hilbert action with the Gauss-Bonnet term  $\mathcal{L}_{GB} = (R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\sigma\tau}R^{\mu\nu\sigma\tau})$ , and a negative cosmological constant  $\Lambda = -\frac{p(p+1)}{2\ell^2}$  in  $p+2$  dimensions can be written as

$$S = \frac{1}{16\pi G} \int_{\mathcal{M}} d^{p+2}x \sqrt{-g} (R - 2\Lambda + \alpha \mathcal{L}_{GB}), \quad (47)$$

where  $\alpha$  is the Gauss-Bonnet coefficient with the same dimension as square of AdS radius  $\ell$ . As  $\mathcal{L}_{GB}$  is nontrivial for  $p \geq 3$ , we will consider the  $p = 3$  case and take the unit AdS radius  $\ell = 1$  in what follows. With the corresponding surface terms [38, 40], we can obtain the equations of motion for the Gauss-Bonnet gravity by varying the action (47) with respect to metric

$$\begin{cases} R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - 6g_{\mu\nu} + \alpha H_{\mu\nu} = 0, \\ H_{\mu\nu} = 2(R_{\mu\sigma\kappa\tau}R_\nu^{\sigma\kappa\tau} - 2R_{\mu\rho\nu\sigma}R^{\rho\sigma} - 2R_{\mu\sigma}R^\sigma_\nu + RR_{\mu\nu}) - \frac{1}{2}g_{\mu\nu}\mathcal{L}_{GB}. \end{cases} \quad (48)$$

The Gauss-Bonnet gravity with a negative cosmological constant is solved by [35, 36] with spherical symmetry. Here we consider the 5-dimensional black brane solution written in the Eddington-Finkelstin coordinates [36, 37]

$$\begin{cases} ds^2 = -r^2 f(r) d\tau^2 + 2d\tau dr + r^2(dx_1^2 + dx_2^2 + dx_3^2) \\ f(r) = \frac{1}{4\alpha} \left( 1 - \sqrt{1 - 8\alpha \left( 1 - \frac{r_h^4}{r^4} \right)} \right). \end{cases} \quad (49)$$

Note that when  $r \rightarrow \infty$ ,  $f(r) \rightarrow \frac{1}{4\alpha} (1 - \sqrt{1 - 8\alpha})$ . One can define the effective  $AdS$  radius  $\ell_e$  in the Gauss-Bonnet gravity as

$$\ell_e^2 = \frac{1 + \sqrt{1 - 8\alpha}}{2} = \frac{4\alpha}{1 - \sqrt{1 - 8\alpha}}. \quad (50)$$

We find that in the case of the Gauss-Bonnet gravity, the correction term (19) and the solution (20) also work with  $p = 3$ . But the form  $F(r)$  has to be replaced by

$$F(r) = \int_r^{r_c} dx \left[ 1 - \frac{r_h^3}{x^3} \frac{1 - 8\alpha}{(1 - 2\alpha [2f(x) + xf'(x)])} \right] \frac{1}{x^2 f(x)}. \quad (51)$$

With the constraint equations  $\partial_i v^i = 0$ , we have checked that (20) solves the equations of motion of the Gauss-Bonnet gravity equations (48) up to order  $\epsilon^2$ . Here when  $r_c \rightarrow \infty$ , the form  $F(r)$  gives the result in [37].

From [39, 40], we can take the Brown-York stress energy tensor for the Gauss-Bonnet gravity as

$$T_{ab} = \frac{1}{8\pi G} \left[ K\gamma_{ab} - K_{ab} - 2\alpha \left( 3J_{ab} - J\gamma_{ab} + 2\hat{P}_{acdb}K^{cd} \right) + C\gamma_{ab} \right], \quad (52)$$

where

$$\hat{P}_{abcd} = \hat{R}_{abcd} + 2\hat{R}_{b[c}\gamma_{d]a} - 2\hat{R}_{a[c}\gamma_{d]b} + \hat{R}\gamma_{a[c}\gamma_{d]b} \quad (53)$$

is an intrinsic tensor associated with the flat induced metric  $\gamma_{ab}$ . So in our case  $\hat{P}_{abcd}$  will not make any contribution to the stress energy tensor of the dual fluid. Namely, we have

$$T_{ab} = \frac{1}{8\pi G} \left[ K\gamma_{ab} - K_{ab} - 2\alpha (3J_{ab} - J\gamma_{ab}) + C\gamma_{ab} \right], \quad (54)$$

with

$$J_{ab} = \frac{1}{3} (2KK_{ac}K_b^c + K_{cd}K^{cd}K_{ab} - 2K_{ac}K^{cd}K_{db} - K^2 K_{ab}). \quad (55)$$

By use of Gauss-Codazzi equations, the conservation of stress energy tensor (54) can also be deduced from the equations (48) of motion of the Gauss-Bonnet gravity [39]. Through some straightforward calculations, the stress energy tensor of the dual fluid to the Gauss-Bonnet gravity is found to be

$$T_{ab} = T_{ab}^{(0)} + T_{ab}^{(1)} + T_{ab}^{(2)} + O(\epsilon^3) \quad (56)$$

where

$$\left\{ \begin{array}{l} T_{ab}^{(0)} dx^a dx^b = \left( -6\sqrt{f_c} - 2C + 8\alpha f_c \sqrt{f_c} \right) r_c^2 f_c d\tau^2 \\ \quad \left( 6\sqrt{f_c} + 2C - 8\alpha f_c \sqrt{f_c} + (1 - 4\alpha f_c) \frac{r_c f'_c}{\sqrt{f_c}} \right) r_c^2 dx_i dx^i \\ T_{ab}^{(1)} dx^a dx^b = -2(1 - 4\alpha f_c) \frac{r_c f'_c}{\sqrt{f_c}} v_i dx^i d\tau \\ T_{ab}^{(2)} dx^a dx^b = (1 - 4\alpha f_c) \left( 3r_c^2 f_c P + v^2 \right) \frac{r_c f'_c}{\sqrt{f_c}} d\tau^2 \\ \quad + (1 - 4\alpha f_c) \frac{r_c f'_c}{\sqrt{f_c}} \left[ \frac{v_i v_j}{r_c^2 f_c} + \left( 1 + \frac{r_c f'_c}{2f_c} \right) r_c^2 P \delta_{ij} \right] dx^i dx^j \\ \quad - [1 - 2\alpha (2f_c + r_c f'_c)] \frac{1 + r_c^2 f_c F'_c}{r_c \sqrt{f_c}} (\partial_i v_j + \partial_j v_i) dx^i dx^j \end{array} \right. \quad (57)$$

The corresponding conservation equations for the stress energy tensor at order  $\epsilon^2$  are

$$\partial^a T_{a\tau} = \frac{(1 - 4\alpha f_c) r_c f'_c}{\sqrt{f_c}} \partial^i v_i = \frac{4}{\sqrt{f_c}} \frac{r_h^4}{r_c^4} \partial_i v^i = 0 \Rightarrow \partial_i v^i = 0, \quad (58)$$

which gives the incompressible condition again, and to the next order

$$\partial^a T_{ai} = \frac{(1 - 4\alpha f_c) f'_c}{r_c f_c \sqrt{f_c}} \left[ \partial_\tau v_i - \frac{\eta_c}{(1 - 4\alpha f_c) f'_c} \frac{f_c}{f'_c} \partial^2 v_i + r_c^2 f_c \left( 1 + \frac{r_c f'_c}{2f_c} \right) \partial_i P + v^j \partial_j v_i \right] = 0, \quad (59)$$

where  $\eta_c = [1 - 2\alpha (2f_c + r_c f'_c)] (1 + r_c^2 f_c F'_c)$ , which will turn out to be the fluid's dynamic viscosity. The equation (59) gives the constraint condition of the solutions of the Gauss-Bonnet gravity (48) at order  $\epsilon^3$ .

In the  $(\tilde{\tau}, \tilde{x}^i)$  coordinates, the corresponding stress energy tensor of the dual fluid is

$$\left\{ \begin{array}{l} \tilde{T}_{ab}^{(0)} d\tilde{x}^a d\tilde{x}^b = \left( -6\sqrt{f_c} - 2C + 8\alpha f_c \sqrt{f_c} \right) d\tilde{\tau}^2 \\ \quad \left( 6\sqrt{f_c} + 2C - 8\alpha f_c \sqrt{f_c} + (1 - 4\alpha f_c) \frac{r_c f'_c}{\sqrt{f_c}} \right) d\tilde{x}_i d\tilde{x}^i \\ \tilde{T}_{ab}^{(1)} d\tilde{x}^a d\tilde{x}^b = -2(1 - 4\alpha f_c) \frac{r_c f'_c}{\sqrt{f_c}} \beta_i d\tilde{x}^i d\tilde{\tau} \\ \tilde{T}_{ab}^{(2)} d\tilde{x}^a d\tilde{x}^b = (1 - 4\alpha f_c) \left( 3P + \beta^2 \right) \frac{r_c f'_c}{\sqrt{f_c}} d\tilde{\tau}^2 \\ \quad + (1 - 4\alpha f_c) \frac{r_c f'_c}{\sqrt{f_c}} \left[ \beta_i \beta_j + \left( 1 + \frac{r_c f'_c}{2f_c} \right) P \delta_{ij} \right] d\tilde{x}^i d\tilde{x}^j \\ \quad - [1 - 2\alpha (2f_c + r_c f'_c)] (1 + r_c^2 f_c F'_c) (\tilde{\partial}_i \beta_j + \tilde{\partial}_j \beta_i) d\tilde{x}^i d\tilde{x}^j, \end{array} \right. \quad (60)$$

from which we can obtain the energy density and pressure of the dual fluid at order  $\epsilon^0$

$$\begin{cases} \rho_0 = -6\sqrt{f_c} - 2C + 8\alpha f_c \sqrt{f_c}, \\ p_0 = 6\sqrt{f_c} + 2C - 8\alpha f_c \sqrt{f_c} + (1 - 4\alpha f_c) \frac{r_c f'_c}{\sqrt{f_c}} \\ \omega_0 \equiv \rho_0 + p_0 = (1 - 4\alpha f_c) \frac{r_c f'_c}{\sqrt{f_c}} = \frac{4}{\sqrt{f_c}} \frac{r_h^4}{r_c^4}, \end{cases} \quad (61)$$

while to order  $\epsilon^2$ , they are

$$\rho_c = \rho_0 + 3\omega_0 P, \quad p_c = p_0 + \omega_0 \left( 1 + \frac{r_c f'_c}{2f_c} \right) P. \quad (62)$$

And the trace of the stress tensor (60) is

$$\tilde{T}_c \equiv \tilde{T}_{ab} \tilde{\gamma}^{ab} = 4 \left[ 3(f_c^{1/2} + f_c^{-1/2}) - 2\alpha f_c^{-3/2} + 2C \right] + \frac{3\omega_0^2}{2\sqrt{f_c}(1 - 4\alpha f_c)} P. \quad (63)$$

If one takes the limit  $r_c \rightarrow \infty$ , some surface counter-terms [40] are needed. Since the intrinsic Riemann tensor associated with  $\gamma_{ab}$  vanishes, in our case, we need only to take  $C = -\frac{2+\sqrt{1-8\alpha}}{\ell_e}$ , which is also consistent with a vanishing trace  $\tilde{T}_c = 0$  when  $r_c \rightarrow \infty$ . Thus we have

$$\lim_{r_c \rightarrow \infty} \rho_c r_c^4 = 3\ell_e r_h^4 (1 + 4P), \quad \lim_{r_c \rightarrow \infty} p_c r_c^4 = \ell_e r_h^4 (1 + 4P). \quad (64)$$

Once again, transforming back to the  $(\tau, x^i)$  coordinates in the boundary metric  $h_{ab}$  (30), the divergent factor  $r_c^4$  could be absorbed. We have checked the result and shown that they match with the stress energy tensor of the dual fluid on the boundary [37].

In this case, we find that the dynamic viscosity

$$\eta_c = [1 - 2\alpha(2f_c + r_c f'_c)] (1 + r_c^2 f_c F'_c) = \frac{1}{16\pi G} \frac{r_h^3}{r_c^3} (1 - 8\alpha). \quad (65)$$

The black brane entropy density associated with the cutoff surface is given by

$$s_c = \frac{1}{4G} \frac{r_h^3}{r_c^3} \implies \frac{\eta_c}{s_c} = \frac{1}{4\pi} (1 - 8\alpha). \quad (66)$$

This ratio of shear viscosity to entropy density is the same as that of the dual fluid to the Gauss-Bonnet gravity on the boundary [9, 10, 11, 12]. Thus, through the hydrodynamic expansion method, we find that the ratio for the dual fluid on the cutoff surface is independent of the cutoff, and does not run with the cutoff [14]. In addition, we find the thermodynamic relation holds and the diffusivity is changed as

$$\omega_0 = T_c s_c, \quad \bar{D}_c = \frac{\eta_c}{\omega_0} T_c = \frac{1}{4\pi} (1 - 8\alpha), \quad (67)$$

where the local temperature on the cutoff surface is

$$T_c = \frac{T_H}{\sqrt{r_c^2 f_c}} = \frac{1}{\sqrt{r_c^2 f_c}} \frac{[r^2 f(r)]'}{4\pi} \Big|_{r=r_h} = \frac{1}{\sqrt{r_c^2 f_c}} \frac{r_h}{\pi}. \quad (68)$$

Finally we give the incompressible Navier-Stokes equations dual to the Gauss-Bonnet gravity

$$\frac{1}{\omega_0} \tilde{\partial}^a \tilde{T}_{ai} = \tilde{\partial}_\tau \beta_i - \nu_c \tilde{\partial}^2 \beta_i + \tilde{\partial}_i P_c + \beta^j \tilde{\partial}_j \beta_i = 0. \quad (69)$$

where

$$\nu_c = \frac{\eta_c}{\omega_0} = \frac{(1 - 8\alpha)}{4\pi T_c}, \quad P_c = \left(1 + \frac{r_c f'_c}{2f_c}\right) P. \quad (70)$$

## 5 Conclusions

By use of the non-relativistic hydrodynamic expansion method, we have solved the equations of motion for Einstein gravity and Gauss-Bonnet gravity with a negative cosmological constant, respectively, and obtained black brane metrics in the region between a finite cutoff surface  $\Sigma_c$  and the black brane horizon, up to order  $\epsilon^2$ . We have calculated the stress energy tensor of the dual fluid on the cutoff surface through the Brown-York tensor on the surface [17]. And then we have discussed some properties of the dual fluid. It turns out that the dual fluid is an incompressible one in both cases, obeys the Navier-Stokes equations, but with different kinematic viscosity. In both cases, the ratio of shear viscosity to entropy density is independent of the cutoff, it means that the ratio does not run with the cutoff surface. When one moves the cutoff surface to spatial infinity, namely takes the limit  $r_c \rightarrow \infty$ , our results can recover those for dual fluids on the boundary of AdS space [27]. In the near horizon limit, our results should also be in contact with the membrane paradigm in [41], where the horizon dynamics is described by incompressible Navier-Stokes equations in the non-relativistic hydrodynamic limit.

Our results presented in this paper together with those in [17, 25, 26] show that the fluids on a finite cutoff surface always obey the incompressible Navier-Stokes equations, for dual gravity solutions up to second order of the non-relativistic hydrodynamic expansion parameter. This may reveal some insights for the holographic dual to asymptotically flat spacetimes. The study of holography on a finite cutoff may also be helpful to understand the microscopic origin of gravity. It is shown in [17] that the entropy flow equation along the radial coordinate is equivalent to a radial Einstein equation. We have checked that this also holds in the Gauss-Bonnet gravity.

Finally we would like to stress that the black brane solutions obtained in this paper are up to second order of the non-relativistic hydrodynamic expansion parameter  $\epsilon$ . In principle, following [26], we can obtain corresponding black brane solutions to an arbitrary order of the expansion parameter. With the resulting solutions, by calculating the corresponding Brown-York tensor in the gravity side, we can obtain the stress energy tensor of dual fluids on the cutoff surface, and then obtain other transport coefficients of the dual fluids.

## Acknowledgements

This work was supported in part by the National Natural Science Foundation of China (No.10821504, No.10975168 and No.11035008), and in part by the Ministry of Science and Technology of China under Grant No. 2010CB833004. LL and YLZ are extremely grateful to Li-Ming Cao, Ya-Peng Hu and Zhang-Yu Nie for calculations by Mathematica, as well as Bin Hu, Hai-Qing Zhang, Huai-Fan Li, Song He, Wei Gu for useful discussions. Finally we would like to thank the two referees for their helpful comments and suggestions.

## References

- [1] J. M. Maldacena, “The large N limit of superconformal field theories and supergravity,” *Adv. Theor. Math. Phys.* **2**, 231 (1998) [*Int. J. Theor. Phys.* **38**, 1113 (1999)] [[arXiv:hep-th/9711200](#)].
- [2] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “Gauge theory correlators from non-critical string theory,” *Phys. Lett. B* **428**, 105 (1998) [[arXiv:hep-th/9802109](#)].
- [3] E. Witten, “Anti-de Sitter space and holography,” *Adv. Theor. Math. Phys.* **2**, 253 (1998) [[arXiv:hep-th/9802150](#)].
- [4] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, *Phys. Rept.* **323**, 183 (2000) [[arXiv:hep-th/9905111](#)].
- [5] G. Policastro, D. T. Son and A. O. Starinets, “The shear viscosity of strongly coupled  $N = 4$  supersymmetric Yang-Mills plasma,” *Phys. Rev. Lett.* **87**, 081601 (2001) [[arXiv:hep-th/0104066](#)].
- [6] P. Kovtun, D. T. Son and A. O. Starinets, “Holography and hydrodynamics: Diffusion on stretched horizons,” *JHEP* **0310**, 064 (2003) [[arXiv:hep-th/0309213](#)].
- [7] A. Buchel and J. T. Liu, “Universality of the shear viscosity in supergravity,” *Phys. Rev. Lett.* **93**, 090602 (2004) [[arXiv:hep-th/0311175](#)].
- [8] P. Kovtun, D. T. Son and A. O. Starinets, “Viscosity in strongly interacting quantum field theories from black hole physics,” *Phys. Rev. Lett.* **94**, 111601 (2005) [[arXiv:hep-th/0405231](#)].
- [9] M. Brigante, H. Liu, R. C. Myers, S. Shenker and S. Yaida, “Viscosity Bound Violation in Higher Derivative Gravity,” *Phys. Rev. D* **77**, 126006 (2008) [[arXiv:0712.0805 \[hep-th\]](#)];
- [10] M. Brigante, H. Liu, R. C. Myers, S. Shenker and S. Yaida, “The Viscosity Bound and Causality Violation,” *Phys. Rev. Lett.* **100**, 191601 (2008) [[arXiv:0802.3318 \[hep-th\]](#)].
- [11] Y. Kats and P. Petrov, “Effect of curvature squared corrections in AdS on the viscosity of the dual gauge theory,” *JHEP* **0901**, 044 (2009) [[arXiv:0712.0743 \[hep-th\]](#)].
- [12] R. G. Cai, Z. Y. Nie, N. Ohta and Y. W. Sun, “Shear Viscosity from Gauss-Bonnet Gravity with a Dilaton Coupling,” *Phys. Rev. D* **79**, 066004 (2009) [[arXiv:0901.1421 \[hep-th\]](#)].

- [13] R. Brustein and A. J. M. Medved, “The Ratio of shear viscosity to entropy density in generalized theories of gravity,” Phys. Rev. D **79**, 021901 (2009) [arXiv:0808.3498 [hep-th]].
- [14] N. Iqbal and H. Liu, ‘Universality of the hydrodynamic limit in AdS/CFT and the membrane paradigm,’ Phys. Rev. D **79**, 025023 (2009) [arXiv:0809.3808 [hep-th]].
- [15] R. G. Cai, Z. Y. Nie and Y. W. Sun, “Shear Viscosity from Effective Couplings of Gravitons,” Phys. Rev. D **78**, 126007 (2008) [arXiv:0811.1665 [hep-th]].
- [16] R. G. Cai, Y. Liu and Y. W. Sun, “Transport Coefficients from Extremal Gauss-Bonnet Black Holes,” JHEP **1004**, 090 (2010) [arXiv:0910.4705 [hep-th]].
- [17] I. Bredberg, C. Keeler, V. Lysov, A. Strominger, “Wilsonian Approach to Fluid/Gravity Duality,” JHEP **1103**, 141 (2011). [arXiv:1006.1902 [hep-th]].
- [18] L. Susskind and E. Witten, “The holographic bound in anti-de Sitter space,” arXiv:hep-th/9805114.
- [19] V. Balasubramanian, P. Kraus, A. E. Lawrence and S. P. Trivedi, “Holographic probes of anti-de Sitter space-times,” Phys. Rev. D **59**, 104021 (1999) [arXiv:hep-th/9808017].
- [20] J. de Boer, E. P. Verlinde and H. L. Verlinde, “On the holographic renormalization group,” JHEP **0008**, 003 (2000) [arXiv:hep-th/9912012].
- [21] D. Nickel and D. T. Son, “Deconstructing holographic liquids,” arXiv:1009.3094 [hep-th].
- [22] I. Heemskerk, J. Polchinski, “Holographic and Wilsonian Renormalization Groups,” [arXiv:1010.1264 [hep-th]].
- [23] T. Faulkner, H. Liu, M. Rangamani, “Integrating out geometry: Holographic Wilsonian RG and the membrane paradigm,” [arXiv:1010.4036 [hep-th]].
- [24] S. J. Sin and Y. Zhou, “Holographic Wilsonian RG Flow and Sliding Membrane Paradigm,” JHEP **1105**, 030 (2011) [arXiv:1102.4477 [hep-th]].
- [25] I. Bredberg, C. Keeler, V. Lysov and A. Strominger, “From Navier-Stokes To Einstein,” [arXiv:1101.2451[hep-th]].
- [26] G. Compere, P. McFadden, K. Skenderis, M. Taylor, “The holographic fluid dual to vacuum Einstein gravity,” [arXiv:1103.3022 [hep-th]].
- [27] S. Bhattacharyya, V. E. Hubeny, S. Minwalla and M. Rangamani, “Nonlinear Fluid Dynamics from Gravity,” JHEP **0802**, 045 (2008) [arXiv:0712.2456 [hep-th]]; M. Rangamani, Class. Quant. Grav. **26**, 224003 (2009) [arXiv:0905.4352 [hep-th]].
- [28] S. Bhattacharyya, S. Minwalla and S. R. Wadia, “The Incompressible Non-Relativistic Navier-Stokes Equation from Gravity,” JHEP **0908** (2009) 059, [arXiv:0810.1545[hep-th]].
- [29] S. Bhattacharyya, R. Loganayagam, I. Mandal, S. Minwalla and A. Sharma, “Conformal Nonlinear Fluid Dynamics from Gravity in Arbitrary Dimensions,” JHEP **0812**, 116 (2008) [arXiv:0809.4272 [hep-th]].

- [30] V. Balasubramanian and P. Kraus, “A stress tensor for anti-de Sitter gravity,” *Commun. Math. Phys.* **208**, 413 (1999) [arXiv:hep-th/9902121].
- [31] R. C. Myers, “Stress tensors and Casimir energies in the AdS/CFT correspondence,” *Phys. Rev. D* **60**, 046002 (1999) [arXiv:hep-th/9903203].
- [32] R. Emparan, C. V. Johnson and R. C. Myers, “Surface terms as counterterms in the AdS/CFT correspondence,” *Phys. Rev. D* **60**, 104001 (1999) [arXiv:hep-th/9903238].
- [33] J. D. Brown and J. W. York, “Quasilocal energy and conserved charges derived from the gravitational action,” *Phys. Rev. D* **47** (1993) 1407, gr-qc/9209012.
- [34] L. Landau and E. Lifshitz, Fluid Mechanics(2nd ed.): Course of Theoretical Physics Vol. 6 , Elsevier Butterworth-Heinemann(1987).
- [35] D. G. Boulware and S. Deser, “String Generated Gravity Models,” *Phys. Rev. Lett.* **55**, 2656 (1985).
- [36] R. G. Cai, “Gauss-Bonnet black holes in AdS spaces,” *Phys. Rev. D* **65**, 084014 (2002) [arXiv:hep-th/0109133].
- [37] Y. P. Hu, H. F. Li and Z. Y. Nie, “The first order hydrodynamics via AdS/CFT correspondence in the Gauss-Bonnet gravity,” *JHEP* **1101**, 123 (2011) [arXiv:1012.0174 [hep-th]]; Y. P. Hu, P. Sun and J. H. Zhang, “Hydrodynamics with conserved current via AdS/CFT correspondence in the Maxwell-Gauss-Bonnet gravity,” *Phys. Rev. D* **83**, 126003 (2011) [arXiv:1103.3773 [hep-th]].
- [38] R. C. Myers, “higher derivative gravity, surface terms and string theory,” *phys. Rev. D* **36**, 392 (1987).
- [39] S. C. Davis, “Generalized Israel junction conditions for a Gauss-Bonnet brane world,” *Phys. Rev. D* **67**, 024030 (2003) [arXiv:hep-th/0208205].
- [40] Y. Brihaye and E. Radu, “Black objects in the Einstein-Gauss-Bonnet theory with negative cosmological constant and the boundary counterterm method,” *JHEP* **0809**, 006 (2008) [arXiv:0806.1396 [gr-qc]]; D. Astefanesei, N. Banerjee and S. Dutta, “(Un)attractor black holes in higher derivative AdS gravity,” *JHEP* **0811**, 070 (2008) [arXiv:0806.1334 [hep-th]].
- [41] C. Eling, I. Fouxon and Y. Oz, “The Incompressible Navier-Stokes Equations From Membrane Dynamics,” *Phys. Lett. B* **680**, 496 (2009) [arXiv:0905.3638 [hep-th]]; C. Eling and Y. Oz, “Relativistic CFT Hydrodynamics from the Membrane Paradigm,” *JHEP* **1002**, 069 (2010) [arXiv:0906.4999 [hep-th]].